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ORIGINAL PAPER

A new coupled wavelet-based method applied to the nonlinear reaction–diffusion equation arising in mathematical chemistry

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Abstract In this paper, we have applied the wavelet-based coupled method for finding the numerical solution of Murray equation. To the best of our knowledge, until now there is no rigorous Legendre wavelets solution has been reported for the Murray equation. The highest derivative in the differential equation is expanded into Legendre series, this approximation is integrated while the boundary conditions are applied using integration constants. With the help of Legendre wavelets operational matrices, the Murray equation is converted into an algebraic system. Block pulse functions are used to investigate the Legendre wavelets coefficient vectors of nonlinear terms. The convergence of the proposed method is proved. Finally, we have given a numerical example to demonstrate the validity and applicability of the method. Moreover the use of proposed wavelet-based coupled method is found to be simple, efficient, less computation costs and computationally attractive.

Keywords Murray equation · Operational matrices · Legendre wavelets · Convergence analysis · Laplace transform method

1 Introduction

Wavelet analysis, as a relatively new and emerging area in applied mathematical Research, has received considerable attention in dealing with PDEs [16–18]. In recent years, wavelet transforms have found their way into many different fields in science

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and engineering [19]. Moreover, wavelet transform methods establish a connection with fast numerical algorithms.

Analytical methods enable researchers to study the effect of different variables or parameters on the function under study easily. Recently, many new approaches to NLPDEs have been proposed, for example, the Adomian decomposition method [21], homotopy analysis method [20,22].

In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based methods, there are two important ways of improving the approximation of the solutions: Increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using various wavelets to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. The basic idea of Legendre wavelet method is to convert the PDEs to a system of algebraic equations by the operational matrices of integral or derivative [10–12]. The main goal is to show how wavelets and multi-resolution analysis can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Razzaghi and Yousefi [10,12] introduced the Legendre wavelet method for solving variational problems and constrained optimal control problems. Hariharan et al. [16–18] had introduced the Haar wavelet method for diffusion equation, convection–diffusion equation, Reaction–diffusion equation, Non linear parabolic equations, fractional Klein-Gordon equations, Sine-Gordon equations and Fisher's equation. Mohammadi and Hosseini [14] had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Parsian [11] introduced two dimension Legendre wavelets and operational matrices of integration. Yousefi [13] applied the Legendre wavelets method for solving differential equations of Lane–Emden type. Maleknejad and Sohrabi [15] established the Legendre wavelets method for solving Fredholm integral equations of the first kind.

In recent years, nonlinear reaction–diffusion equations (NLRDEs) have been widely studied and applied in biological science and engineering [1–4]. This study concerns the numerical solutions of nonlinear reaction–diffusion modelling the dynamics of diffusion and nonlinear reproduction for a population [5,6,8,9,22]. The associated nonlinear reaction–diffusion equation was initiated by Fisher [3] to describe the propagation behaviour of a virile mutant. The nonlinear reaction–diffusion equations describe a population of diploid individuals [1,3].

In this work, we have applied a wavelet-based coupled method (LLWM) which combines the Laplace transform method and the Legendre wavelets method for the numerical solution of Murray equation.

This paper is organized as follows: Basic definitions of wavelets, Legendre wavelets and their properties are described in Sect. 2. Then, the method of solution of the Murray equation by the LLWM is presented in Sect. 3. In Sect. 4, the convergence analysis is described. In Sect. 5, several numerical examples are presented to demonstrate the effectiveness of the proposed method. Concluding remarks are given in Sect. 6.

2 Legendre wavelets and properties

2.1 Wavelets

Wavelets are the family of functions which are derived from the family of scaling function $\{\emptyset_{j,k:}k \in Z\}$ where:

$$\emptyset(x) = \sum_k a_k \emptyset(2x - k) \tag{2.1}$$

For the continuous wavelets, the following equation can be represented:

$$\Psi_{a,b}(x) = |a|^{\frac{-1}{2}} \Psi\left(\frac{x-b}{a}\right) \quad a, b \in R, a \neq 0. \tag{2.2}$$

where a and b are dilation and translation parameters, respectively, such that $\Psi(x)$ is a single wavelet function.

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$\mathrm{a} = \mathrm{a}_0^{-\mathrm{j}}, \quad \mathrm{a}_0 > 1, \ \mathrm{b}_0 > 1, \tag{2.3}$$

$$b = kb_0 a_0^{-j}, \quad j, k \in Z. \tag{2.4}$$

Then, a family of discrete wavelets can be constructed as follows:

$$\Psi_{j,k} = |a_0|^{\frac{1}{2}} \Psi(2^j x - k), \tag{2.5}$$

So, $\Psi_{j,k}(x)$ constitutes an orthonormal basis in L^2 (R), where $\Psi(x)$ is a single function.

2.2 Legendre wavelets

The Legendre wavelets are defined by

$$\Psi_{nm}(t) = \begin{cases} \sqrt{m + \frac{1}{2}}\, 2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \text{for } \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

where $m = 0, 1, 2, \ldots, M - 1$ and $k = 1, 2, \ldots, 2^{j-1}$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality, then, the wavelets $\Psi_{nm}(t)$ form an orthonormal basis for L^2 [0,1]. In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$\begin{aligned} p_0 &= 1, \\ p_1 &= x, \\ p_{m+1}(x) &= \frac{2m+1}{m+1} x\, p_m(x) - \frac{m}{m+1} p_{m-1}(x). \end{aligned} \tag{2.7}$$

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval [0,1]. Note that, in the general form of Legendre wavelets, the dilation parameter is $a = 2^{-j}$ and the translation parameter is $b = n2^j$ [10].

2.3 Two-dimensional Legendre wavelets

Two-dimensional Legendre wavelets in $L^2(R)$ over the interval $[0,1] \times [0,1]$ as the form

$$\Psi_{n,m,n',m'}(x,y) = \begin{cases} \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)} & 2^{\frac{k+k'}{2}} p_m(x) p_{m'}(y), \\ \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, & \frac{n'-1}{2^{k-1}} \le y \le \frac{n'}{2^{k'-1}}; \\ 0, & otherwise. \end{cases} \tag{2.8}$$

and m = 0, 1, 2, ..., $M' - 1$, m' = 0, 1, 2, 3, ..., $M' - 1$, n = 1, 2, ..., 2^{k-1}, n' = 1, 2, ..., $2^{k'-1}$

$$\text{where} P_m(x) = \overline{P_{m'}}(2^k x - 2n + 1),\ P_{m'}(y) = \overline{P_{m'}}(2^{k'} y - 2n' + 1), \tag{2.9}$$

$\overline{P_m}$ are Legendre functions of order m defined over the interval [−1,1].

By using two-dimensional shifted Legendre polynomials into $x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and $y \in \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]$, the $\int_0^1 \Psi_{n,m,n',m'}(x,y)$ can be written as

$$\int_0^1 \Psi_{n,m,n',m'}(x,y) = A_{m,m'} \cdot P_{m'}(x) P_{m'}(y) \chi_{\left[\substack{\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}}\right]}(x,y), \tag{2.10}$$

in which $A_{m,m'} = \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)} 2^{\frac{k+k'}{2}}$ and $\chi_{\left[\substack{\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}}\right]}(x,y)$ is a characteristic function defined as $\chi_{\left[\substack{\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}}\right]}(x,y) = \begin{cases} 1, & x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], y \in \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]; \\ 0, & otherwise \end{cases}$

Two dimension Legendre Wavelets are an orthonormal set over [0,1] × [0,1].

$$\int_0^1\int_0^1 \Psi_{n,m,n',m'}(x,y)\,\Psi_{n_1,m_1,n'_1,m'_1}(x,y)\,dxdy = \delta_{n,n_1}\delta_{n',n'_1}\delta_{m',m'_1} \tag{2.11}$$

The function u(x,y) $\in L^2(R)$ defined over [0,1] × [0,1] may be expanded as

$$\text{u(x, y)} = \text{X(x)Y(y)} \cong \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} c_{n,m,n',m'}\Psi_{n,m,n',m'}(x,y) \tag{2.12}$$

If the infinite series in Eq. (2.12) is truncated, then Eq. (2.13) can be written as

$$\text{u (x, y)} = \text{X(x)Y(y)} \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} c_{n,m,n',m'}\Psi_{n,m,n',m'}(x,y) \tag{2.13}$$

where $c_{n,m,n',m'} = \int_0^1\int_0^1 X(x)Y(y)\Psi_{n,m,n',m'}(x,y)dxdy$.

The Eq. (2.13) can be expressed as the form

$$u(x,y) = c^T\cdot\Psi(x,y) \tag{2.14}$$

where C and Ψ(x,y) are coefficients matrix and wavelets vector matrix respectively. The number of dimensions of C and Ψ(x,y) are $2^{k-1}2^{k'-1}$ MM′ × 1, and given by

$$\begin{aligned} C = [&c_{1,0,1,0},\ldots,c_{1,0,1,M'-1},c_{1,0,2,0},\ldots,c_{1,0,2,M'-1},\ldots,c_{1,0,2^{k'-1},0},\ldots,\\ &c_{1,0,2^{k'-1},M'-1},\ldots,c_{1,M-1,1,0},\ldots,c_{1,M-1,1,M'-1},c_{1,M-1,2,0},\ldots,\\ &c_{1,M-1,2,M'-1},\ldots,c_{1,M-1,2^{K-1},0},\ldots,c_{1,M-1,2^{K-1},M'-1},\ldots,c_{2,0,1,0},\ldots,\\ &c_{2,0,1,M'-1},c_{2,0,2,0},\ldots,c_{2,0,2,M'-1},\ldots,c_{2,0,2^{K-1},0},\ldots,c_{2,0,2^{k-1},M'-1},\ldots,\\ &c_{2,M-1,1,0},\ldots,c_{2,M-1,1,M'-1},c_{2,M-1,2.0},\ldots,c_{2,M-1,2,M'-1},\ldots,\\ &c_{2,M-1,2^{k-1},0},\ldots,c_{2,M-1,2^{k-1},M'-1},\ldots,c_{2^{k-1},0,1,0},\ldots,c_{2^{k-1},0,1M'-1},\\ &c_{2^{k-1},0,2,0},\ldots,c_{2^{k-1},0,M'-1},\ldots,c_{2^{k-1},0,2^{k-1},0},\ldots,c_{2^{k-1},M-1,2^{k'-1},M'-1}]^T \end{aligned} \tag{2.15}$$

$$\begin{aligned} \Psi = [&\Psi_{1,0,1,0},\ldots,\Psi_{1,0,1,M'-1},\Psi_{1,0,2,0},\ldots,\Psi_{1,0,2^{k-1},0},\ldots,\Psi_{1,0,2^{k'-1},M'-1},\ldots,\\ &\Psi_{1,M-1,1,0},\ldots,\Psi_{1,M-1,1,M'-1},\Psi_{1,M-1,2,0},\ldots,\Psi_{1,M-1,2,M'-1},\ldots,\\ &\Psi_{1,M-1,2^{k-1},0},\ldots,\Psi_{1,M-1,2^{k-1},M'-1},\ldots,\Psi_{2,0,1,0},\ldots,\Psi_{2,0,1,M'-1},\\ &\Psi_{2,0,2,0},\ldots,\Psi_{2,0,2,M'-1},\ldots,\Psi_{2,0,2^{k'-1},0},\ldots,\Psi_{2,0,2^{k-1},M'-1},\ldots,\\ &\Psi_{2,M-1,1,0},\ldots,\Psi_{2,M-1,1,M'-1},\Psi_{2,M-1,2,0},\ldots,\Psi_{2,M-1,2,M'-1},\ldots,\\ &\Psi_{2,M-1,2^{k'-1},0},\ldots,\Psi_{2,M-1,2^{k'-1},M'-1},\Psi_{2^{k-1},0,1,0},\ldots,\Psi_{2^{k-1},0,1,M'-1},\\ &\Psi_{2^{k-1},0,2,0},\ldots,\Psi_{2^{k-1},0,2,M'-1},\ldots,\Psi_{2^{k-1},0,2^{k-1},0},\ldots,\\ &\Psi_{2^{k-1},M-1,2^{k-1},M'-1}]^T \end{aligned} \tag{2.16}$$

The integration of the product of two Legendre wavelet function vectors is obtained as

$$\int_0^1 \int_0^1 \Psi(x, y)\, \Psi^T(x, y)\, dxdy = I \tag{2.17}$$

where I is the identity matrix.

Another form of the two dimensional Legendre wavelets by using the one dimensional Legendre wavelets was given in [10].

A two-dimensional function f(x,y) defined [0,1) × [0,1) may be expanded by Legendre wavelet series as

$$f(x, y) = \sum_{i=1}^{2^k M} \sum_{j=1}^{2^k M} C_{ij} \Psi_{\mathrm{i}}(\mathrm{x}) \Psi_{\mathrm{j}}(\mathrm{y}) = \Psi^{\mathrm{T}}(\mathrm{x})\, \mathrm{C} \Psi(\mathrm{y}) \tag{2.18}$$

where

$$C_{ij} = \int_0^1 f(x, y)\, \Psi_{\mathrm{i}}(\mathrm{x})\, \mathrm{dx} \int_0^1 \mathrm{f}(\mathrm{x}, \mathrm{y}) \Psi_{\mathrm{j}}(\mathrm{y})\, \mathrm{dt} \tag{2.19}$$

Equation (2.18) can be written into the discrete form (in matrix form) by

$$f(x, y) = \Psi^{\mathrm{T}}(\mathrm{x})\, \mathrm{C} \Psi(\mathrm{y}) \tag{2.20}$$

Where C and Ψ(t) are $2^{k-1} M x 1$ matrices given by

$$\mathrm{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2^{k-1}M} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2^{k-1}M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2^{k-1}M,0} & c_{2^{k-1}M,1} & \cdots & c_{2^{k-1}M 2^{k-1}M} \end{bmatrix}$$

The two dimensional Legendre wavelet operational matrix of integration has been derived in Refs. [12,10].

Theorem 1 *Let* $\Psi(x, y)$ *be the two-dimensional Legendre wavelets vector defined in Eq. (2.8), we have*

$$\frac{\partial \Psi(x, y)}{\partial x} = D_x \Psi(x, y) \tag{2.21}$$

where D_x *is* $2^{k-1,}2^{k'-1}\,MM' \times 2^{k-1}2^{k'-1}\,MM'$ *and has the form as follows:*

$$D_x = \begin{bmatrix} D & O' & \dots & 0' \\ 0' & D & \dots & 0' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \dots & D \end{bmatrix}$$

In which $0'$ *and* D *is* $2^{k-1}2^{k'-1}\,MM' \times 2^{k-1}2^{k'-1}\,MM'$ *matrix and the element of D is defined as follows:*

$$D_{r,s} = \begin{cases} 2^k\sqrt{(2r-1)(2s-1)}I, & r = 2, 3, \dots, M;\; s = 1, \dots, r-1;\; r+s \text{ is odd} \\ 0, & otherwise \end{cases} \tag{2.22}$$

and I, O are $2^{k'-1}M' \times 2^{k'-1}M'$ *identity matrices.*

Theorem 2 *Let* $\Psi(x, y)$ *be the two-dimensional Legendre wavelets vector defined in Eq. (2.23), we have*

$$\frac{\partial \Psi(x, y)}{\partial x} = D_y \Psi(x, y), \tag{2.23}$$

$$D_y = \begin{bmatrix} D & O' & \dots & 0' \\ 0' & D & \dots & 0' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \dots & D \end{bmatrix},$$

where D_y *is* $2^{k-1,}2^{k'-1}\,MM' \times 2^{k-1}2^{k'-1}\,MM'$ *and* O', *D is* $MM' \times MM'$ *matrix is given as*

$$D = \begin{bmatrix} F & O & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & F \end{bmatrix},$$

in which O and F is $M' \times MM'$ *matrix, and F is defined as follows:*

$$F_{r,s} = \begin{cases} 2^{k'}\sqrt{(2r-1)(2s-1)}, & r = 2, \dots, M';\; S = 1, \dots, r-1;\; and\, r+s \text{ is odd} \\ 0, & otherwise \end{cases} \tag{2.24}$$

By using Eqs. (2.21) *and* (2.23)*, the operational matrices for nth derivative can be derived as*

$$\frac{\partial^n \Psi(x,y)}{\partial x^n} = D_x^n \Psi(x,y), \frac{\partial^m \Psi(x,y)}{\partial y^m} = D_y^m \Psi(x,y)$$
$$\frac{\partial^{n+m} \Psi(x,y)}{\partial x^n \partial y^m} = D_x^n D_y^m \Psi(x,y)$$

where D^n is the nth power of matrix D.

2.4 Block pulse functions (BPFs)

The block pulse functions form a complete set of orthogonal functions which defined on the interval [0, b) by

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m}b \le t < \frac{i}{m}b, \\ 0, & elsewhere \end{cases} \tag{2.25}$$

for i = 1, 2, . . ., m. It is also known that for any absolutely integrable function f(t) on [0,b) can be expanded in block pulse functions:

$$f(t) \cong \xi^T B_m(t) \tag{2.26}$$
$$\xi^T = [f_1, f_2, \ldots, f_m], B_m(t) = [b_1(t), b_2(t), \ldots, b_m(t)] \tag{2.27}$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{b} \int_0^b f(t) b_i(t)\, dt \tag{2.28}$$

Remark 1 Let A and B are two matrices of m x m, then $A \otimes B = (a_{ij} \times b_{ij})_{mm}$.

Lemma 1 *Assuming f(t) and g(t) are two absolutely integrable functions, which can be expanded in block pulse function as f(t) = FB(t) and g(t) = GB(t) respectively, then we have*

$$f(t)\, g(t) = FB(t)\, B^T(t)\, G^T = HB(t) \tag{2.29}$$

where H = F ⊗G.

2.5 Approximating the nonlinear term

The Legendre wavelets can be expanded into m-set of block-pulse Functions as

$$\Psi(t) = \emptyset_{m\times m} B_m(t) \tag{2.30}$$

Taking the collocation points as following

$$t_i = \frac{i - 1/2}{2^{k-1}M}, \quad i = 1, 2, \ldots, 2^{k-1}M \tag{2.31}$$

The m-square Legendre matrix $\emptyset_{m\times m}$ is defined as

$$\emptyset_{m\times m} \cong [\Psi\,(t_1)\,\Psi\,(t_2)\ldots\Psi(t_{2^{k-1}M})] \tag{2.32}$$

The operational matrix of product of Legendre wavelets can be obtained by using the properties of BPFs, let $f(x,t)$ and $g(x,t)$ are two absolutely integrable functions, which can be expanded by Legendre wavelets as $f(x,t) = \Psi^T(x)F\Psi(t)$ and $g(x,t) = \Psi^T(x)G\Psi(t)$ respectively.

From Eq. (2.30), we have

$$f\,(x,t) = \Psi^T\,(x)\,F\Psi\,(t) = B^T\,(x)\,\emptyset_{mm}^T F\emptyset_{mm}B\,(t)\,, \tag{2.33}$$

$$g\,(x,t) = \Psi^T\,(x)\,G\Psi\,(t) = B^T\,(x)\,\emptyset_{mm}^T G\emptyset_{mm}B\,(t)\,, \tag{2.34}$$

and $F_b = \emptyset_{mm}^T F\emptyset_{mm}$, $G_b = \emptyset_{mm}^T G\emptyset_{mm}$, $H_b = F_b \otimes G_b$.

Then,

$$\begin{aligned} f\,(x,t)\,g\,(x,t) &= B^T H_b B\,(t)\,,\\ &= B^T\,(x)\,\emptyset_{mm}^T inv\left(\emptyset_{mm}^T\right) H_b inv\left(inv\left(\emptyset_{mm}^T\right) H_b inv\,(\emptyset_{mm})\right)\emptyset_{mm}B\,(t)\\ &= \Psi^T\,(x)\,H\Psi\,(t) \end{aligned} \tag{2.35}$$

where H $= inv(\emptyset_{mm}^T)H_b inv((\emptyset_{mm}))$

2.6 Function approximation

A given function f(x) with the domain [0,1] can be approximated by:

$$\text{f(x)} = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} c_{k,m}\Psi_{k,m}\,(x) = C^T.\Psi\,(x)\,. \tag{2.36}$$

Here C and Ψ are the matrices of size $(2^{j-1}M \times 1)$.

$$\mathrm{C}=[\mathrm{c}_{1,0}, \mathrm{c}_{1,1}, \ldots, \mathrm{c}_{1,\mathrm{M}-1}, c_{2,0}, \mathrm{c}_{2,1}, \ldots, \mathrm{c}_{2,\mathrm{M}-1}, \ldots, \mathrm{c}_2^{\mathrm{j}-1}, 1, \ldots, \mathrm{c}_2^{\mathrm{j}-1}{}_{,\mathrm{M}-1}]^{\mathrm{T}} \tag{2.37}$$

$$\Psi\,(x) = [\Psi_{1,0}, \Psi_{1,1}, \Psi_{2,0}, \Psi_{2,1}, \ldots, \Psi_{2,M-1}, \ldots, \Psi_{2^{j-1},M-1}]^{\mathrm{T}}. \tag{2.38}$$

3 Mathematical model and the method of solution

Consider the NLRDEs with convection term of the form [7]

$$\frac{\partial U}{\partial t} = A(U)\frac{\partial^2 U}{\partial x^2} + B(U)\frac{\partial U}{\partial x} + C(U), \quad 0 \leq x < 1, 0 \leq t < 1. \tag{3.1}$$

where $U(x,t)$ is an unknown function, $A(U)$, $B(U)$ and $C(U)$ are arbitrary smooth functions. Equation (3.1) is a well-known nonlinear second order evolution equation describing various models in biology [7].

When $A(U) = 1$, $B(U) = \mu_1 U$ and $C(U) = \mu_2 U - \mu_3 U^2$

Here μ_1, μ_2,$\mu_3 \in R$.

Equation (3.1) becomes

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \mu_1 U \frac{\partial U}{\partial x} + \mu_2 U - \mu_3 U^2, \quad 0 \leq x < 1, 0 \leq t < 1, \tag{3.2}$$

which is called the nonlinear Murray equation with initial condition:

$$U(x,0) = f(x), \quad 0 \leq x < 1 \tag{3.3}$$

and mixed boundary conditions

$$\left.\begin{array}{r} U(0,t) = G(t), \quad 0 \leq t < 1 \\ \frac{\partial U}{\partial x} = I(t), \quad 0 \leq t < 1 \end{array}\right\} \tag{3.4}$$

The exact solution for Eq. (3.2) is given by

$$U(x,t) = \frac{\mu_2 + c_1 e^{(\gamma^2 t + \gamma x)}}{\mu_3 + c_0 e^{-(\mu_2 t)}},$$

where $\gamma = \frac{\mu_3}{\mu_1} and\ \mu_1 \neq 0$

c_0 is a constant such that $\mu_3 + c_0 e^{(-\lambda^2 t)} \neq 0$ and c_1 is an arbitrary constant.

Taking Laplace transform on both sides of Eq. (3.2), we get

$$sL(U) - U(x,0) = L[U_{xx} + \mu_1 U U_x + \mu_2 U - \mu_3 U^2] \tag{3.5}$$

$$sL(U) = U(x,0) + L[U_{xx} + \mu_1 U U_x + \mu_2 U - \mu_3 U^2] \tag{3.6}$$

$$L(U) = \frac{U(x,0)}{s} + \frac{1}{s} L[U_{xx} + \mu_1 U U_x + \mu_2 U - \mu_3 U^2] \tag{3.7}$$

Taking inverse Laplace transform to Eq. (3.7) we get

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s} L[U_{xx} + \mu_1 U U_x + \mu_2 U - \mu_3 U^2]\right) \tag{3.8}$$

Because

$$L^{-1}\left[\frac{1}{s}L(t^n)\right] = L^{-1}\left(\frac{n!}{s^{n+2}}\right)$$
$$= \frac{1}{n+1}t^{n+1}; \quad (n = 0, 1, 2, \ldots) \tag{3.9}$$

We have

$$L^{-1}[s^{-1}L()] = \int_0^t (.)dt \tag{3.10}$$

From Eq. (3.8)

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L\left(U_{xx} + g\left(U\right)\right)\right) \tag{3.11}$$

where $g\left(U\right) = \mu_1 U U_x + \mu_2 U - \mu_3 U^2$

By using the Legendre wavelets method,

$$\left.\begin{array}{l} U(x,t) = C^T\psi(x,t) \\ U(x,0) = S^T\psi(x,t) \\ g(U) = G^T\psi(x,t) \end{array}\right\} \tag{3.12}$$

Substituting Eq. (3.12) in Eq. (3.11), we obtain

$$C^T = S^T + (C^T Dx^2 + G^T)P_t^2 \tag{3.13}$$

Here G^T has a nonlinear relation with C. When we solve a nonlinear algebraic system, we get the solution is more complex and large computation time. In order to overcome the above drawbacks, we introduce an approximation formula as follows:

$$U_{n+1} = U(x,0) + \Pi\left[\frac{\partial^2 U_n}{\partial x^2} + g(U_n)\right] \tag{3.14}$$

where $g\left(U\right) = \mu_1 U U_x + \mu_2 U - \mu_3 U^2$

Expanding u(x,t) by Legendre wavelets using the following relation

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 + G_n^T\right] P_t^2 \tag{3.15}$$

4 Convergence analysis

$$U^* = U_0 + \Pi\left[U^*_{xx} + g(U^*)\right] \tag{4.1}$$

$$U_{n+1} = U_0 + \Pi\left[(U_n)_{xx} + g(U_n)\right] \tag{4.2}$$

Subtracting Eq. (4.1) from Eq. (4.2), we obtain

$$U_{n+1} - U^* = \Pi[(U_n - U^*)_{xx} + g(U_n) - g(U^*)] \tag{4.3}$$

Using Lipschitz condition,
$\|g(U_n) - g(U^*)\| \leq \gamma \|U_n - U^*\|$, we have

$$\left\|U_{n+1} - U^*\right\| \leq \left\|\Pi(U_n - U^*)_{xx}\right\| + \left\|\Pi(g(U_n) - g(U^*))\right\| \tag{4.4}$$

$$\leq \left\|\Pi\left(U_n - U^*\right)_{xx}\right\| + \gamma\left\|\Pi\left(U_n - U^*\right)\right\| \tag{4.5}$$

Let $U_{n+1} = C^T_{n+1}\psi(x, t)$

$$U^* = C^T\psi(x, t)$$

$$\in^T_{n+1} = C^T_{n+!} - C^T$$

Equation (4.5) gives

$$\in^T_{n+1} \leq \in^T_n \left\|D^2_x P^2_t + \gamma P^2_t\right\| \tag{4.6}$$

The following formula Eq. (4.7) can be obtained by using recursive relation.

$$\in^T_{n+1} \leq \in^T_n \left\|D^2_x P^2_t + \gamma P^2_t\right\|^n \in_0 \tag{4.7}$$

When $Lim_{n\to\infty}\left\|D^2_x P^2_t + \gamma P^2_t\right\|^n$=0, the series solution of Eq. (3.2) using the LLWM converges to $u^*(x)$. By using the definitions of D_x and P_t, we can get the value of γ.
Suppose $k = k' = 1$ and $M = M'$, the maximum element of D_x and P_t is $2\sqrt{(2M - 1)(2M - 3)}$ and 0.5 respectively.

5 Illustrative example

Example 5.1 Consider the Murray equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + U\frac{\partial U}{\partial x} + U - U^2, \quad 0 \leq x < 1, 0 \leq t < 1 \tag{5.1}$$

With the initial condition

$$U(x,0) = \frac{\mu_2 + c_1 e^{(\gamma x)}}{\mu_3 + c_0} \tag{5.2}$$

and mixed boundary conditions

$$U(0,t) = \frac{\mu_2 + c_1 e^{(\gamma^2 t)}}{\mu_3 + c_0 e^{(-\mu_2 t)}} \tag{5.3}$$

$$\frac{\partial U(0,t)}{\partial x} = \frac{c_1 \gamma e^{(\gamma^2 t)}}{\mu_3 + c_0 e^{(-\mu_2 t)}} \tag{5.4}$$

with $c_0 = 1, c_1 = 1$ *and* $\gamma = 1$.

We start the first iteration; an initial guess of the solution of u_0 is required. We select $u_0 = u(x,0)$, and expanding u by the Legendre wavelets, we gain

The Legendre wavelets scheme for Eq. (5.1) is given by

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 + G_n^T \right] P_t^2$$

Our proposed method (LLWM) can be compared with Cherniha's results [7]. Good agreement with the exact solution is observed.

More efficient and accurate results can be obtained by using larger values of M. Comparison with these algorithms shows that the LLWM is competitive and efficient.

The numerical solutions of Murray's equation (Example.5.1) for different values of x and t is presented in Table 1. Our LLWM results are in excellent agreement with the exact solution and those obtained by the Cherniha's method [7].

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel ∼1596 Mhz.

Table 1 A comparison between the exact and the LLWM for various values of (x, t) and $M = 4$

x	t	Exact solution $U(x,t)$	Numerical solution u_{LLWM}
0.125	0.125	1.21329571	1.21329569
0.125	0.875	2.62430764	2.62430765
0.375	0.125	1.40702557	1.40702556
0.375	0.875	3.16921683	3.16921680
0.625	0.125	1.65577963	1.65577962
0.625	0.875	3.86889407	3.86889405
0.875	0.125	1.97518616	1.97518615
0.875	0.875	4.76729744	4.76729740

6 Conclusion

In this work, a new coupled wavelet-based method has been successfully employed to obtain the numerical solutions of Murray's equation arising in mathematical biology. The proposed scheme is the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. This method shows higher efficiency than the traditional Legendre wavelet method for solving nonlinear PDEs. Numerical example illustrates the powerful of the proposed scheme LLWM. Also this paper illustrates the validity and excellent potential of the LLWM for nonlinear PDEs. The numerical solutions obtained using the proposed method show that the solutions are in very good coincidence with the exact solution. In addition the calculations involved in LLWM are simple, straight forward and low computation cost. In Sect. 4, we have developed the convergence of the proposed algorithm.

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